Image Transforms

Digital Image Processing

Fundamentals of Digital Image Processing, A. K. Jain

- 2D Orthogonal and Unitary Transform:
  - Orthogonal Series Expansion:

$$v(k,l)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}u(m,n)a_{k,l}(m,n)\quad 0\le k,l\le N-1$$

$$u(m,n)=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}v(k,l)a_{k,l}^{*}(m,n)\quad 0\le m,n\le N-1$$

  - $\{a_{k,l}(m,n)\}$: a set of complete orthonormal basis:
  - Orthonormality: $\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}a_{k,l}(m,n)a_{k',l'}^{*}(m,n)=\delta(k-k',l-l')$
  - Completeness: $\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}a_{k,l}(m,n)a_{k,l}^{*}(m',n')=\delta(m-m',n-n')$

- 2D Orthogonal and Unitary Transform:
  - $v(m,n)$: Transformed coefficients
  - $\mathbf{V}=\{v(m,n)\}$: Transformed Image
  - Orthonormality requires:

$$u_{P,Q}(m,n)=\sum_{k=0}^{P-1}\sum_{l=0}^{Q-1} v(k,l)\,a_{k,l}^{*}(m,n) \quad P\le N, Q\le N$$

$$u_{P,Q}(m,n)=\arg\min_{\hat{u}}\left\{\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\left[u(m,n)-\hat{u}(m,n)\right]^{2}\right\}$$

- Separable Unitary Transform:
  - Computational Complexity of former: $O(N^4)$
  - With Separable Transform: $O(N^3)$

$$a_{k,l}(m,n) = a_k(m)b_l(n) \triangleq a(k,m)b(l,n)$$

$$a_k(m), b_l(n): \text{One Dimensional Complete Orthonormal Basis}$$

  - Orthonormality and Completeness:
    - $\mathbf{A}=\{a(k,m)\}$ and $\mathbf{B}=\{b(l,n)\}$ are unitary:
  - Usually **B** is selected same as **A** (**A=B**):

$$A\left(A^*\right)^T = A^T A^* = I$$

  - Unitary Transform!

- Separable Unitary Transform:

$$v(k,l)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}a(k,m)u(m,n)a(l,n)\Leftrightarrow \mathbf{V}=\mathbf{AUA}^{\mathbf{T}}=\left(\mathbf{A}\left[\mathbf{AU}\right]^{\mathbf{T}}\right)^{\mathbf{T}}$$

$$u(m,n)=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}a^{*}(k,m)v(m,n)a^{*}(l,n)\Leftrightarrow \mathbf{U}=\left(\mathbf{A}^{*}\right)^{\mathbf{T}}\mathbf{VA}^{*}$$

  – Basis Images:

$$\left(\mathbf{A}^{*}\right)^{\mathbf{T}}=\left[\cdots\middle|\quad \mathbf{a}_{k}^{*}\quad\middle|\cdots\right]: k^{th}\ Column$$

$$\mathbf{A}_{\mathbf{k,l}}^{*}=\mathbf{a}_{k}^{*}\left(\mathbf{a}_{l}^{*}\right)^{T}$$

$$\mathbf{U}=\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}v(k,l)\mathbf{A}_{\mathbf{k,l}}^{*}:\ \text{Linear Combination of } \mathrm{N}^{2}\ \text{matrices}$$

- Properties of Unitary Transform:

$$\mathbf{v} = \mathbf{A}\mathbf{u} \Rightarrow \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2$$

$$\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\left|u(m,n)\right|^2 = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1}\left|v(k,l)\right|^2$$

- Two Dimensional Fourier Transform:

$$v(k,l)=\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}u(m,n)W_N^{km}W_N^{ln}, \quad W_N \triangleq \exp\left(\frac{-j2\pi}{N}\right)$$

$$u(m,n)=\frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}v(k,l)W_N^{-km}W_N^{-ln}$$

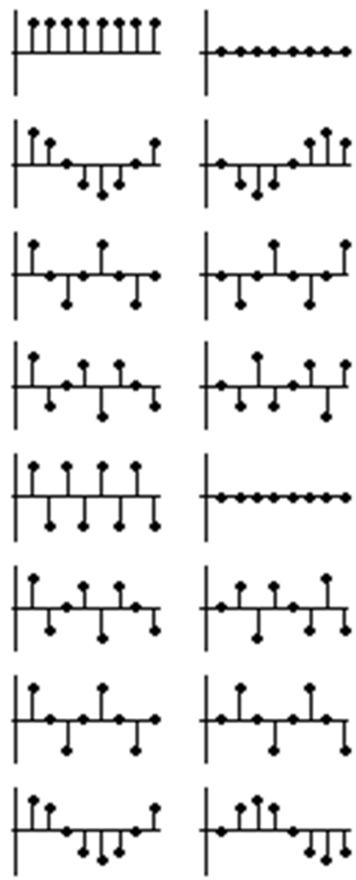
$$\left.\begin{aligned} v(k,l)&=\frac{1}{N}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1}u(m,n)W_N^{km}W_N^{ln} \\ u(m,n)&=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1}v(k,l)W_N^{-km}W_N^{-ln}\end{aligned}\right\}\text{Unitary DFT Pair}$$

- Matrix Notation: $\mathbf{V}=\mathbf{FUF}\Leftrightarrow\mathbf{U}=\mathbf{F}^*\mathbf{VF}^*$

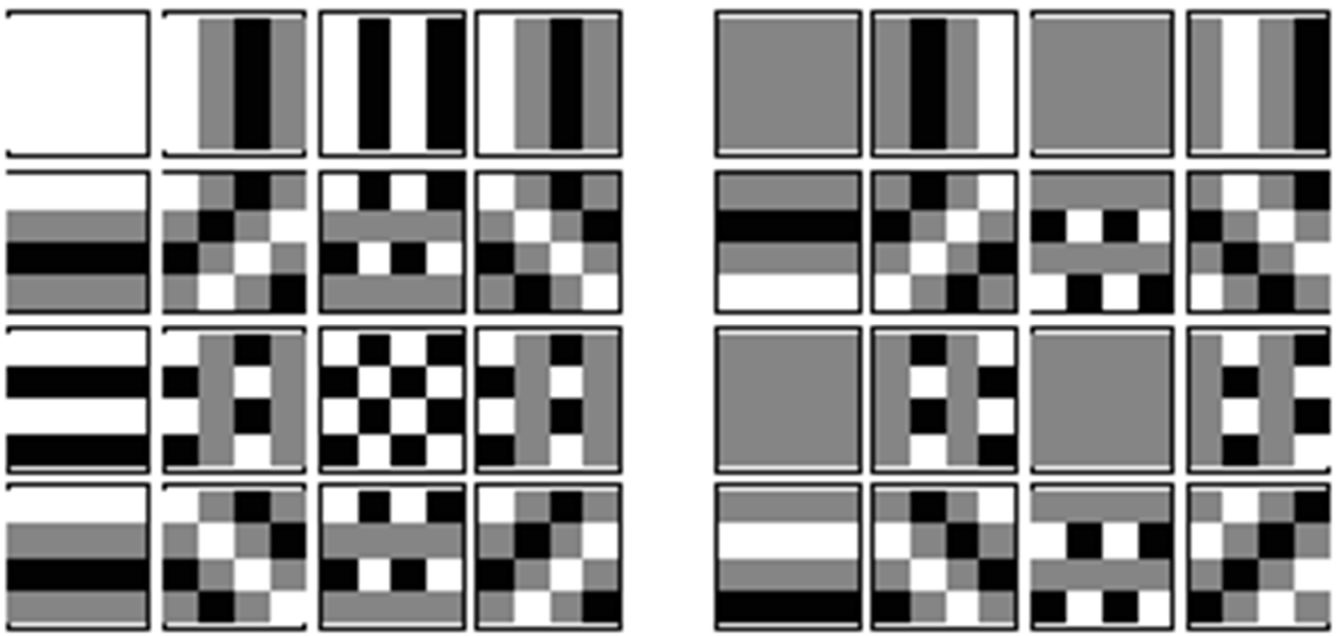
$$\mathbf{F}=\left\{\frac{1}{\sqrt{N}}W_N^{kn}\right\}_{k,n=0}^{N-1}$$

- DFT Properties:
  - Symmetric Unitary
  - Periodic Extension
  - Sampled Fourier
  - Fast
  - Conjugate Symmetry
  - Circular Convolution

- Basis of DFT (Real and Imaginary):



- Basis of DFT (Real and Imaginary):



- Discrete Cosine Transform (DCT):
  - 1D Cases:

$$\mathbf{C} = \{c(k,n)\}$$

$$c(k,n) = \begin{cases} \dfrac{1}{\sqrt{N}} & k = 0,\ 0 \le n \le N-1 \\ \dfrac{2}{\sqrt{N}} \cos\left(\dfrac{\pi(2n+1)k}{2N}\right) & 1 \le k \le N-1,\ 0 \le n \le N-1 \end{cases}$$

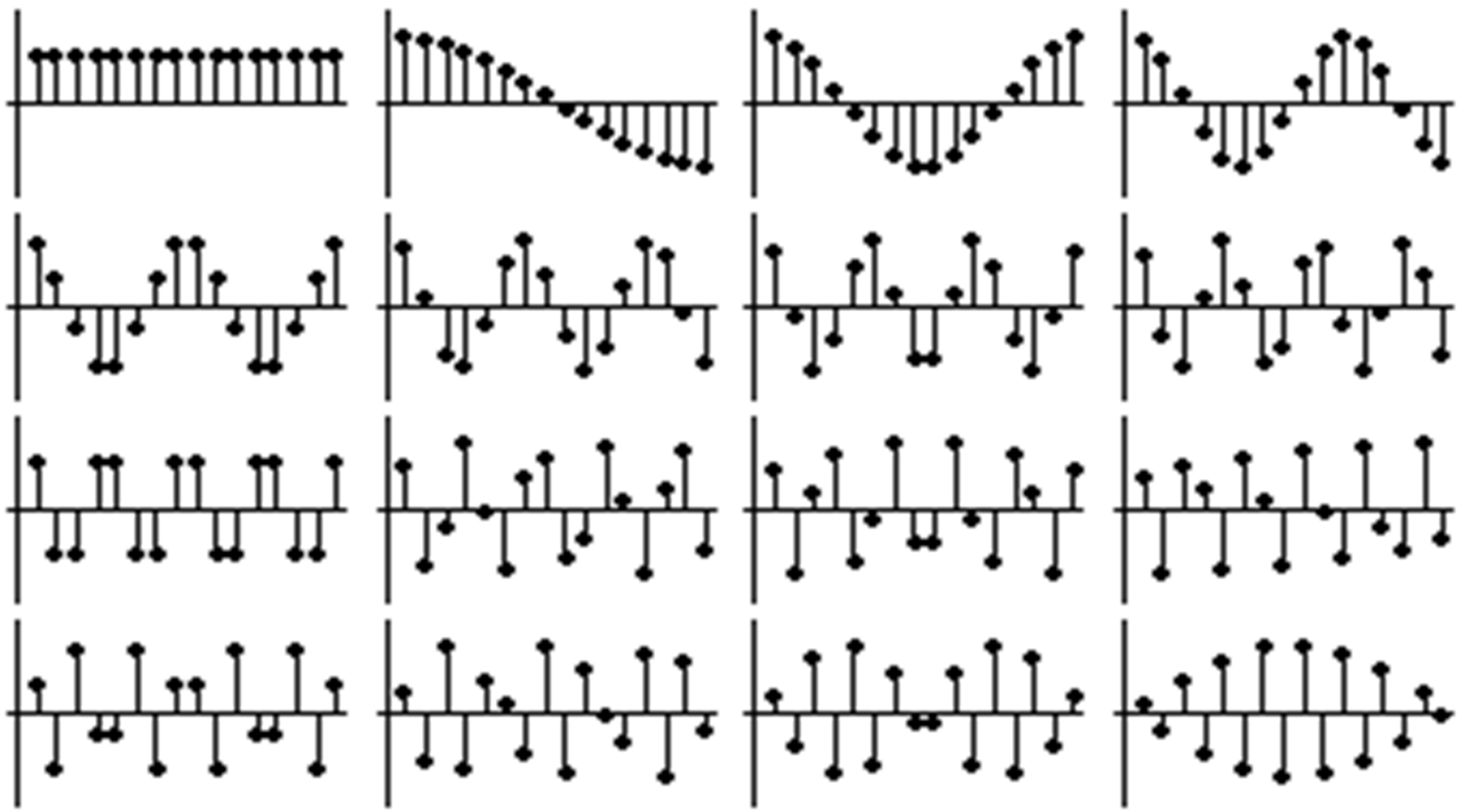
$$v(k) = \alpha(k) \sum_{n=0}^{N-1} u(n) \cos\left(\frac{\pi(2n+1)k}{2N}\right), \quad 0 \le k \le N-1$$

$$\alpha(0) \triangleq \frac{1}{\sqrt{N}}, \quad \alpha(k) \triangleq \frac{2}{\sqrt{N}}, \quad 1 \le k \le N-1$$

$$u(n) = \sum_{k=0}^{N-1} \alpha(k) v(k) \cos\left(\frac{\pi(2n+1)k}{2N}\right), \quad 0 \le n \le N-1$$

- Properties of DCT:
  - Real and Orthogonal: $\mathbf{C}=\mathbf{C}^* \rightarrow \mathbf{C}^{-1}=\mathbf{C}^T$
  - Not! Real part of DFT
  - Fast Transform
  - Excellent Energy compaction (Highly Correlated Data)
- Two Dimensional Cases:
  - $\mathbf{A}=\mathbf{A}^*=\mathbf{C}$

- DCT Basis:



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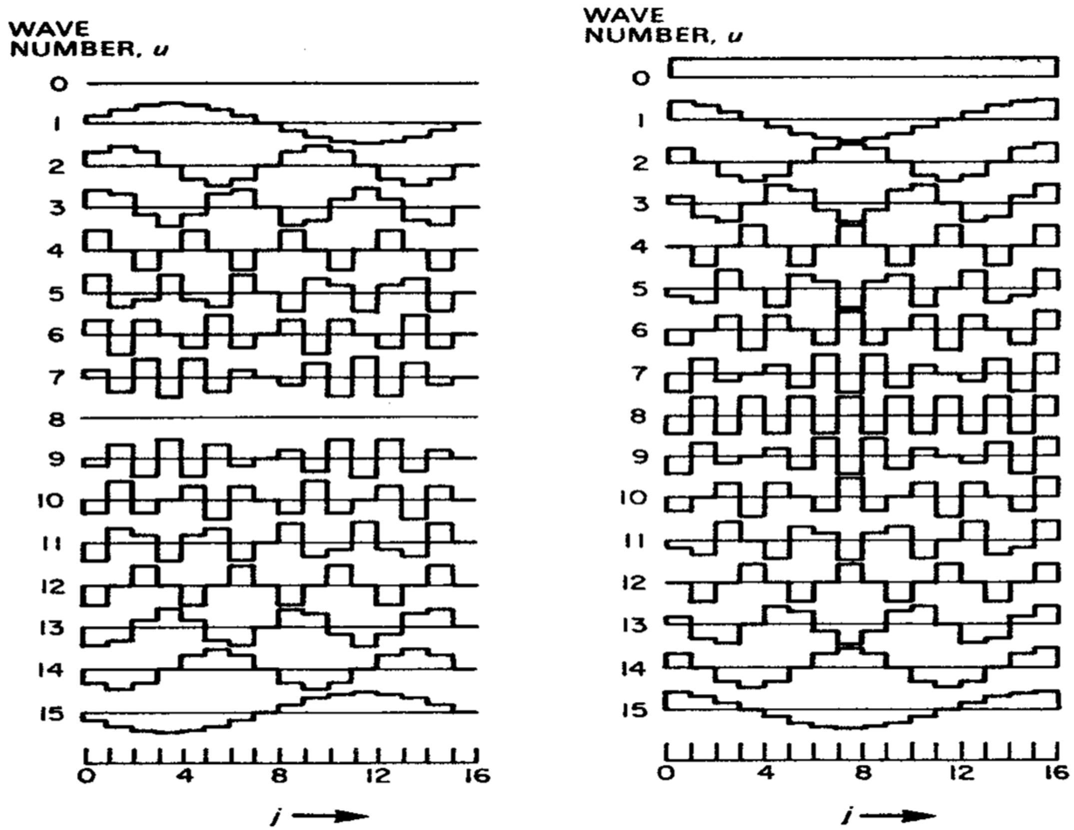
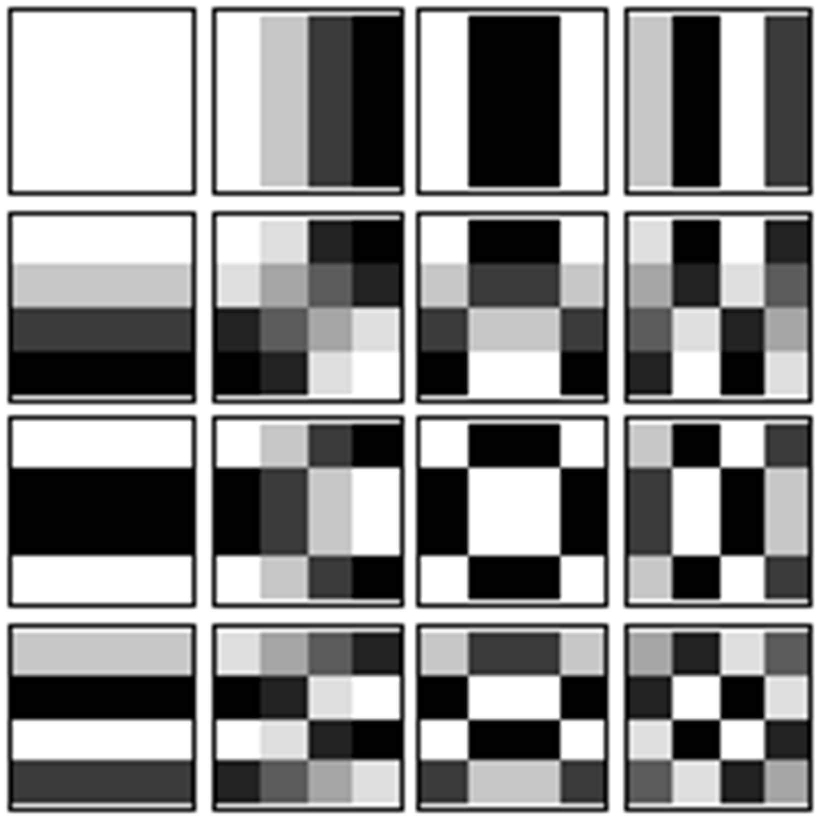


(a) Sine component (b) Cosine component

- DCT Basis:



- Discrete Sine Transform (DST):
  - 1D Cases:

$$\boldsymbol{\psi} = \{\psi(k,n)\}$$

$$\psi(k,n) = \sqrt{\frac{2}{N+1}} \sin\left(\frac{\pi(n+1)(k+1)}{N+1}\right), \quad 0 \le k,n \le N-1$$

$$v(k) = \sqrt{\frac{2}{N+1}} \sum_{n=0}^{N-1} u(n) \sin\left(\frac{\pi(n+1)(k+1)}{N+1}\right), \quad 0 \le k \le N-1$$

$$u(n) = \sqrt{\frac{2}{N+1}} \sum_{k=0}^{N-1} v(k) \sin\left(\frac{\pi(n+1)(k+1)}{N+1}\right), \quad 0 \le n \le N-1$$

  - 2D Case: $\mathbf{A}=\mathbf{A}^*=\mathbf{A}^T=\boldsymbol{\Psi}$

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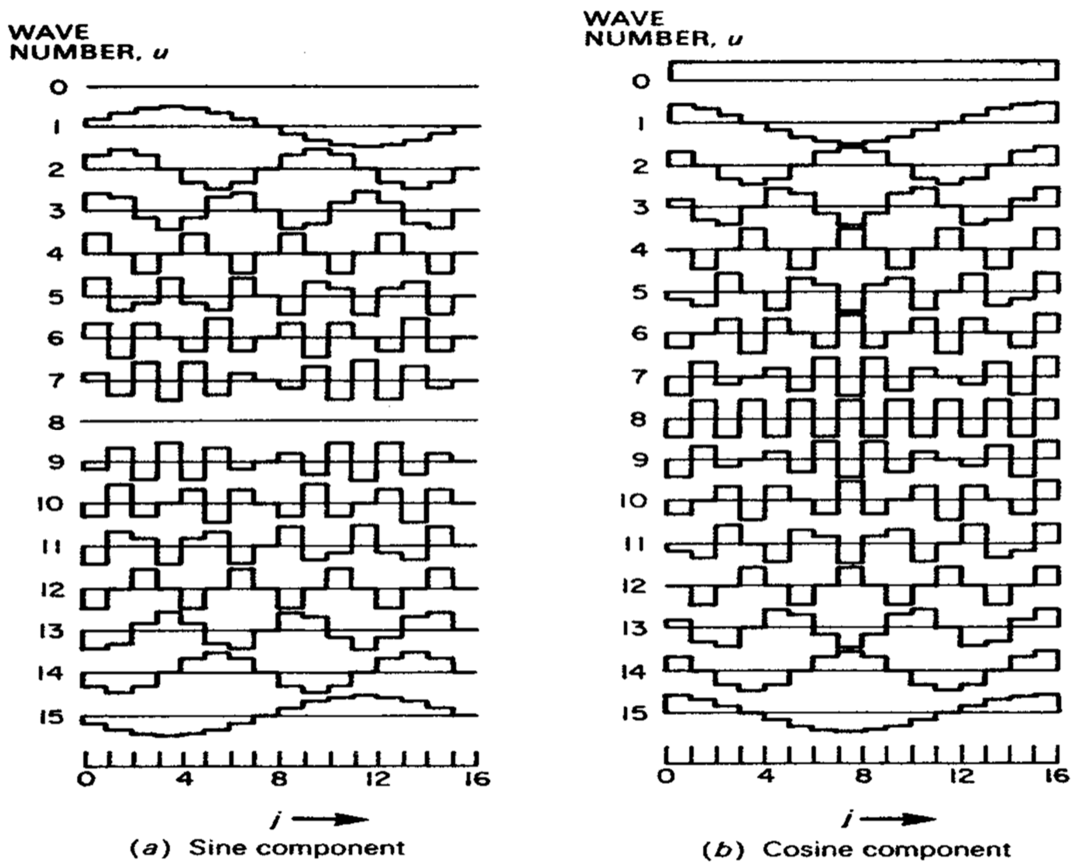

- Properties of DST:
  - Real, Symmetric and Orthogonal: $\mathbf{\Psi} = \mathbf{\Psi}^{*} = \mathbf{\Psi}^{T} = \mathbf{\Psi}^{-1}$
  - Forward and Inverse are identical
  - Not! Imaginary part of DFT
  - Fast Transform

- Welsh-Hadamard Transform (WHT): $N=2^n$
  - 1D Cases:

$$H_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_n = H_{n-1} \otimes H_1 = H_1 \otimes H_{n-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} H_{n-1} & H_{n-1} \\ H_{n-1} & -H_{n-1} \end{bmatrix}$$

Walsh Function

$$H_3 = \frac{1}{\sqrt{8}}\begin{array}{c} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{bmatrix} \end{array}\begin{array}{c} 0 \\ 7 \\ 3 \\ 4 \\ 1 \\ 6 \\ 2 \\ 5 \end{array}$$

- Welsh-Hadamard Transform (WHT): $N=2^n$

$$\mathbf{v} = \mathbf{H}\mathbf{u} \Leftrightarrow \mathbf{u} = \mathbf{H}\mathbf{v}, \quad \mathbf{H} = \mathbf{H}_{\mathbf{n}}, N = 2^n$$

$$v(k) = \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} u(m)(-1)^{b(k,m)} \quad 0 \le k \le N-1$$

$$u(m) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} v(k)(-1)^{b(k,m)} \quad 0 \le m \le N-1$$

$$b(k,m) = \sum_{i=0}^{n-1} k_i m_i \quad k_i, m_i = 0, 1$$

$$k = \sum_{i=0}^{n-1} k_i 2^i, \quad m = \sum_{i=0}^{n-1} m_i 2^i$$

  - 2D Cases: $\mathbf{A}=\mathbf{A}^*=\mathbf{A}^T=\mathbf{H}$

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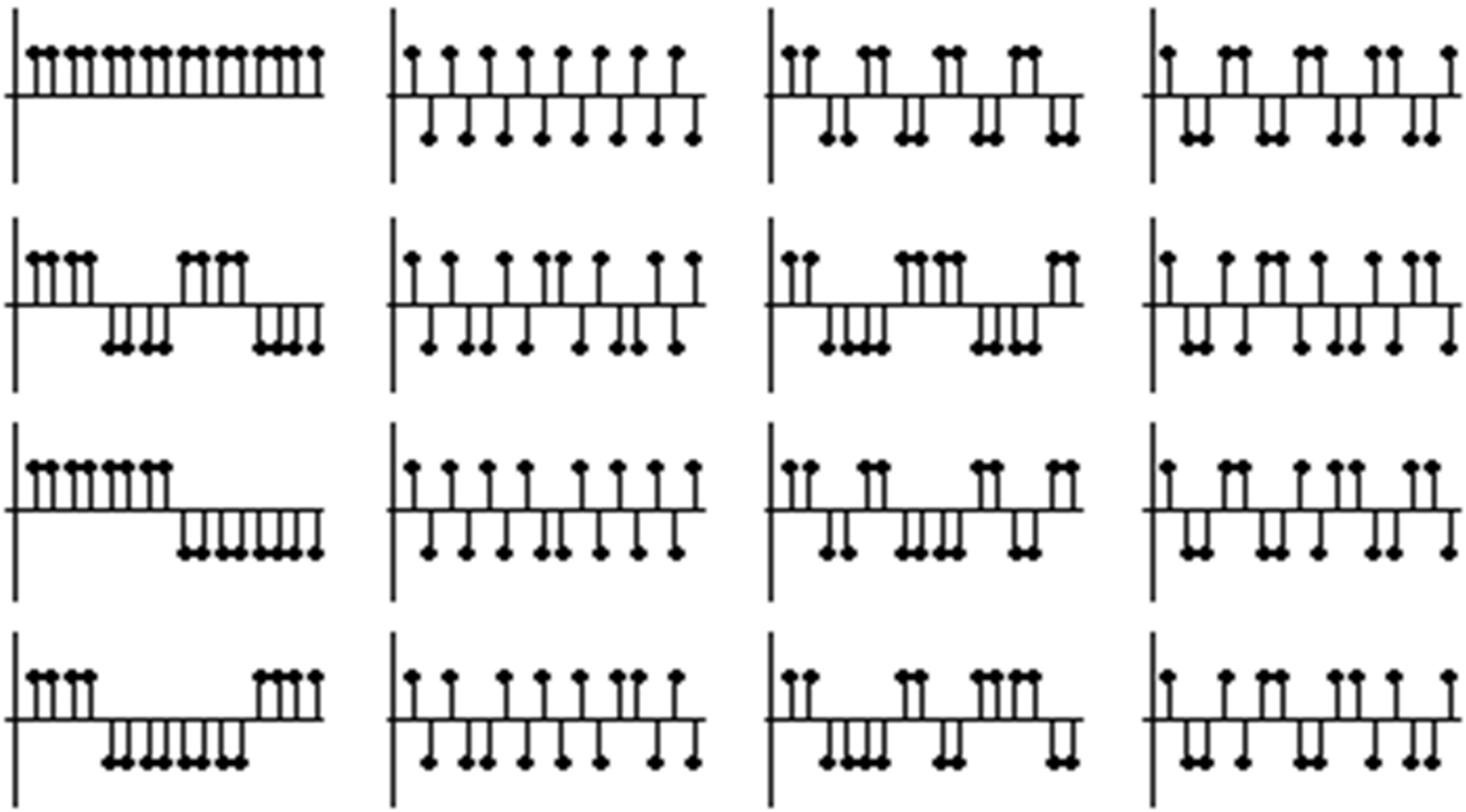
- Welsh-Hadamard Transform Properties:
  - Real, Symmetric, and orthogonal: $\mathbf{H}=\mathbf{H}^{*}=\mathbf{H}^{T}=\mathbf{H}^{-1}$
  - Ultra Fast Transform (±1)
  - Good-Very Good energy compactness

- Welsh-Hadamard Basis:



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- Welsh-Hadamard Basis

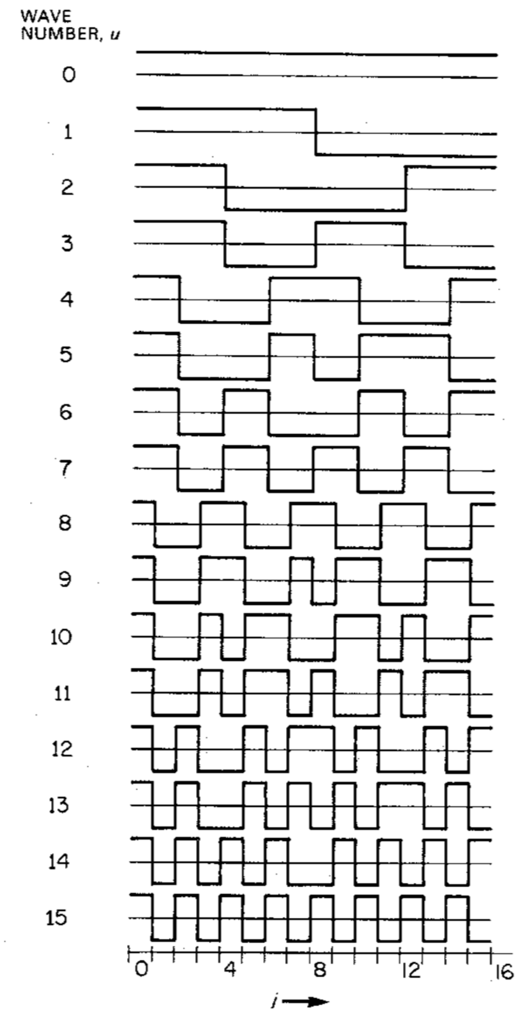
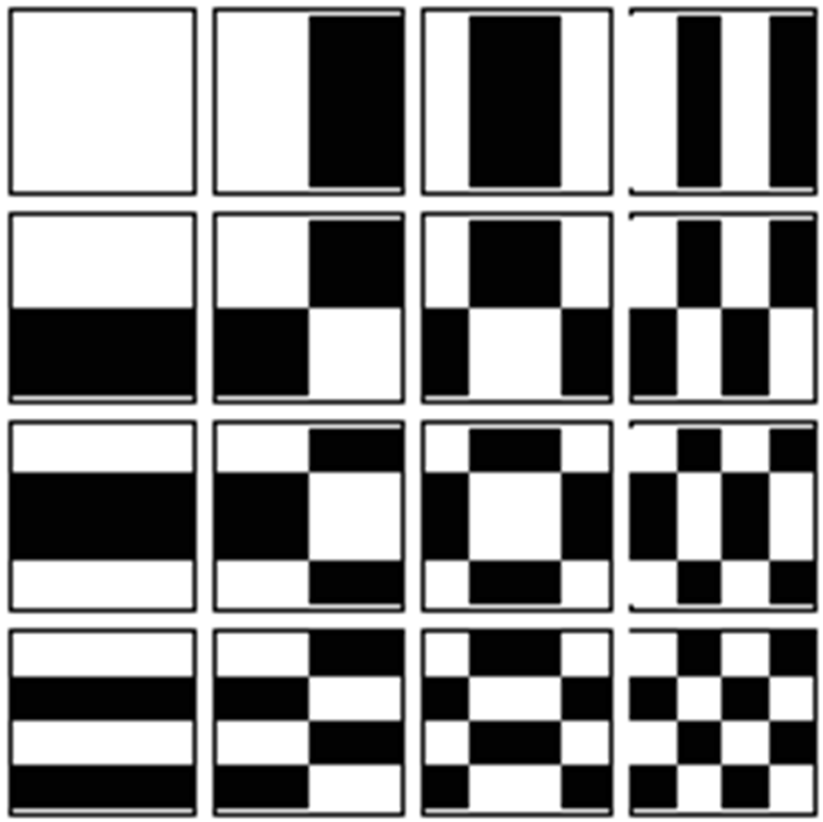


FIGURE 8.4-2. Hadamard transform basis functions, $N = 16$.

- Welsh-Hadamard Basis



- Haar Transform $N=2^n$
  - 1D Cases:

$$h_k(x), \quad x \in [0,1], \quad k = 0, \dots, N-1$$

$$k = 2^p + q - 1$$

$$\begin{cases} 0 \le p \le n-1; q = 0,1 & p = 0 \\ 1 \le q \le 2^p & p \ne 0 \end{cases}$$

$$h_0(x) \triangleq h_{0,0}(x) = \frac{1}{\sqrt{N}}, \quad x \in [0,1]$$

$$h_k(x) \triangleq h_{p,q}(x) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2} & \frac{q-1}{2^p} \le x \le \frac{q-1/2}{2^p} \\ -2^{p/2} & \frac{q-1/2}{2^p} \le x \le \frac{q}{2^p} \\ 0 & \text{O.W.} \end{cases}$$

$$x = \frac{m}{N}, \quad m = 0, 1, \dots, N-1$$

- Haar Transform $N=2^n$
  - 1D Cases:

$$
\mathbf{H}_r = \frac{1}{\sqrt{8}}
\begin{array}{c}
\left[\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
\sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -2
\end{array}\right]
\end{array}
\begin{array}{c}
0 \\ 1 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2
\end{array}
$$

  - 2D Cases: $\mathbf{Hr*A*Hr^T}$

Image Transforms

- Haar Basis Function

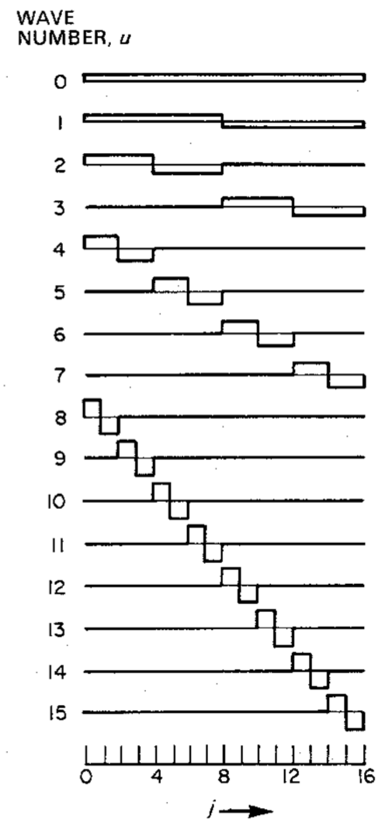
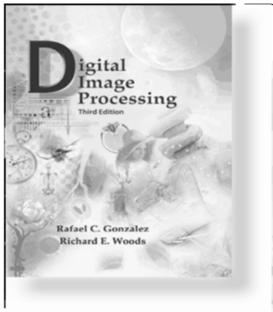


FIGURE 8.4-4. Haar transform basis functions, $N = 16$.



- Haar Transform Properties
  - Real and Orthogonal: $\mathbf{Hr}=\mathbf{Hr}^{*}$, $\mathbf{Hr}^{-1}=\mathbf{Hr}^{T}$
  - Fast Transform
  - Poor energy compactness

- Slant Transform ($N=2^n$)

$$S_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 1 & 0 \\ a_2 & b_2 & -a_2 & b_2 \\ 0 & 1 & 0 & -1 \\ -b_2 & a_2 & b_2 & a_2 \end{bmatrix}\begin{bmatrix} S_1 & 0_2 \\ 0_2 & S_1 \end{bmatrix}$$

- Slant Transform ($N=2^n$)

$$S_n = \frac{1}{2^{\frac{1}{2}}}\left[\begin{array}{cc|c|cc|c} 1 & 0 & & 1 & 0 & \\ a_n & b_n & 0 & -a_n & b_n & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{(N/2)-2} & \multicolumn{2}{c|}{0} & I_{(N/2)-2} \\ \hline 0 & 1 & & 0 & -1 & \\ -b_n & a_n & 0 & b_n & a_n & 0 \\ \hline \multicolumn{2}{c|}{0} & I_{(N/2)-2} & \multicolumn{2}{c|}{0} & -I_{(N/2)-2} \end{array}\right]\left[\begin{array}{c|c} S_{n-1} & 0 \\ \hline 0 & S_{n-1} \end{array}\right]$$

$$N = 2^n, \quad a_{n+1} = \left(\frac{3N^2}{4N^2-1}\right)^{\frac{1}{2}}, \quad b_{n+1} = \left(\frac{N^2-1}{4N^2-1}\right)^{\frac{1}{2}}$$

- Slant Basis Function

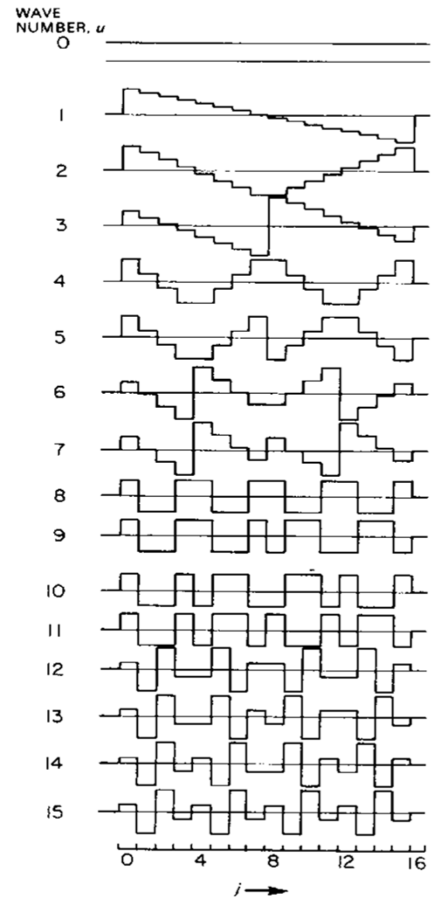


FIGURE 8.4-5. Slant transform basis functions, $N = 16$.

- Slant Transform Properties:
  - Real and Orthogonal $S=S^{*}$ $S^{-1}=S^{T}$
  - Fast
  - Very Good Compactness